MATH 134A+105A+110A Review: Lagrange Multiplier Method

Facts to Know

To find the absolute/global maximum and minimum values of $f(x,y)$ subject to the constraint $g(x,y) = k$

(a) Find all values of x, y, λ such that

$$\begin{cases} \nabla f(x,y) = \lambda \nabla g(x,y) \\ g(x,y) = k \end{cases}$$

(b) Evaluate f at all the points (x,y) that result from the previous step. The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Examples

1. Maximize $f(x,y) = x + y$ subject to the constraint $x^2 + y^2 = 1$.

(a) solve for x, y, λ

$$\begin{cases} \nabla f = \lambda \nabla g \\ g(x,y) = k \end{cases}$$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ x^2 + y^2 = 1 \end{cases}$$

$$\begin{cases} 1 = \lambda (2x) \\ 1 = \lambda (2y) \\ x^2 + y^2 = 1 \end{cases}$$

$$x = \frac{1}{2\lambda}, \quad y = \frac{1}{2\lambda}$$

$$2\left(\frac{1}{2\lambda}\right)^2 = 1$$

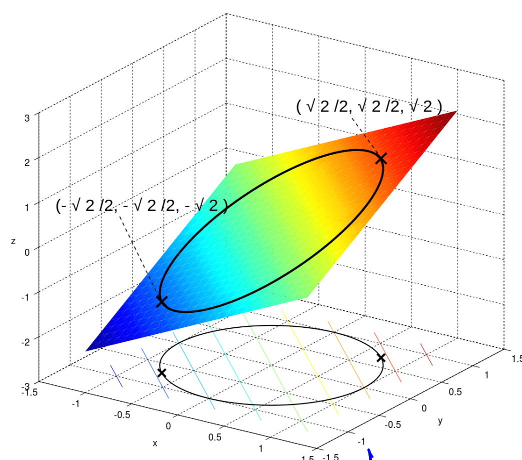
$$\frac{1}{2} \frac{1}{\lambda^2} = 1$$

$$\frac{1}{2} = \lambda^2$$

$$\lambda = \pm\sqrt{1/2}$$

$$x = \frac{1}{2\lambda} = \frac{1}{2(\pm\sqrt{1/2})}$$

$$y = \frac{1}{2(\pm\sqrt{1/2})}$$



	x	y
$\lambda = +\sqrt{1/2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$\lambda = -\sqrt{1/2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$

Candidate points

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, $\left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$

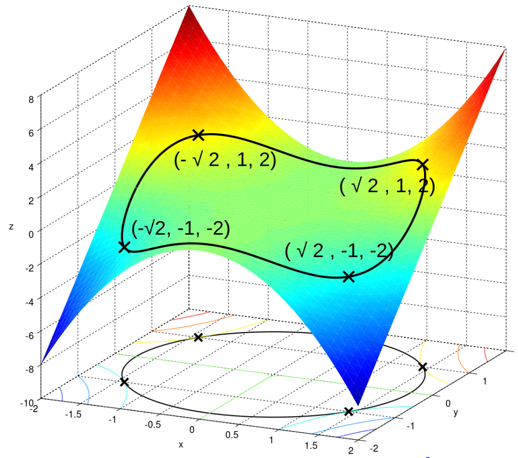
MAX MIN

$\sqrt{2}$ $-\sqrt{2}$ (values)

2. Maximize $f(x,y) = x^2 y$ subject to the constraint $x^2 + y^2 - 3 = 0$.

$f_x = 2xy \qquad g_x = 2x$

$f_y = x^2 \qquad g_y = 2y$



$$\begin{cases} 2xy = \lambda\, 2x \\ x^2 = \lambda\, 2y \\ x^2 + y^2 - 3 = 0 \end{cases} \qquad 2x(y-\lambda) = 0$$

Case 1 : $x = 0$ $\qquad 0 = \lambda \cdot 2 \cdot y$

$y^2 - 3 = 0$

$y = \pm\sqrt{3}$

$\lambda = 0$

$(0, +\sqrt{3}) \overset{f}{\mapsto} 0$

$(0, -\sqrt{3}) \overset{f}{\mapsto} 0$

Case 2: $y - \lambda = 0$

$y = \lambda$

$x^2 = 2\lambda^2$

$x = \pm\sqrt{2}\,\lambda$

$2\lambda^2 + \lambda^2 - 3 = 0$

$3\lambda^2 - 3 = 0$ $\qquad \lambda = \pm 1$

$(\sqrt{2}, 1) \overset{f}{\mapsto} 2$

$(-\sqrt{2}, 1) \overset{f}{\mapsto} 2$

$(\sqrt{2}, -1) \overset{f}{\mapsto} -2$

$(-\sqrt{2}, -1) \overset{f}{\mapsto} -2$